Mixed Partitions of $\mathrm{PG}(3, q^2)$

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A mixed partition of $\mathrm{PG}(2n-1, q^2)$ is a partition of the points of $\mathrm{PG}(2n-1, q^2)$ into $(n-1)$-spaces and Baer subspaces of dimension $2n-1$. In [6] it is shown that such a mixed partition of $\mathrm{PG}(2n-1, q^2)$ can be used to construct a $(2n-1)$-spread of $\mathrm{PG}(4n-1, q)$ and hence a translation plane of order q^{2n}. In this paper we provide several new examples of such mixed partitions in the case when $n = 2$.

Key Words: partitioning, spreads, Baer subspaces

1. INTRODUCTION

Let $\mathrm{PG}(d, q)$ represent the projective geometry of dimension d over the finite field $GF(q)$. An $(n-1)$-spread of $\mathrm{PG}(2n-1, q)$ is a collection of $q^n + 1$ mutually disjoint $(n-1)$-spaces that together partition the point set of $\mathrm{PG}(2n-1, q)$. By a result of Bruck and Bose [1], the study of finite affine translation planes is equivalent to the study of such spreads.

In [6], a method is given where a partition of $\Pi = \mathrm{PG}(2n-1, q^2)$, $n \geq 2$, is used to construct a $(2n-1)$-spread of $\Sigma_0 = \mathrm{PG}(4n-1, q)$. The partition is made up of α copies of $\mathrm{PG}(2n-1, q)$, that we call *Baer subspaces*, and β copies of $\mathrm{PG}(n-1, q^2)$. Unfortunately, few examples of such *mixed* partitions are given in [6]. We provide several new examples here.

2. A MIXED PARTITION FROM A SINGER GROUP

We will describe our first example of a mixed partition using group theory. The end result will be a partition of $\Pi \cong \mathrm{PG}(2n-1, q^2)$ with exactly

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2 distinct $(n-1)$-spaces and all of the remaining points partitioned into Baer subspaces. We build the partition by considering one of its Baer subspaces $B_0 \cong \mathrm{PG}(2n-1,q)$. Consider B_0 modelled by $L = GF(q^{2n})$, a $2n$-dimensional vector space over $F = GF(q)$, and let η be a primitive element of L. Then we can think of the points of B_0 as the field elements

$$1, \eta, \eta^2, ..., \eta^{q^{2n-1}+\cdots+q^2+q}.$$

Multiplication by η on these field elements induces a cyclic collineation that acts regularly on the points of $\mathrm{PG}(2n-1,q)$, a *Singer* group. Let Θ represent this Singer group.

THEOREM 2.1. *The orbits of* Θ *form a mixed partition of* $\mathrm{PG}(2n-1,q^2)$ *containing two copies of* $\mathrm{PG}(n-1,q^2)$ *and* $(q-1)(q^{2n-2}+q^{2n-4}+\cdots+q^2+1)$ *Baer subspaces.*

Proof. It is not hard to see that a generator for Θ can be represented by a block diagonal matrix. These blocks determine the two orbits that form the $(n-1)$-spaces of our mixed partition, and the remaining orbits are all of full length, namely $q^{2n-1}+\cdots+q+1$. See [9] for a detailed proof that these orbits form Baer subspaces.

The total number of points of $\mathrm{PG}(2n-1,q^2)$ is $q^{4n-2}+q^{4n-4}+\cdots+q^2+1$, and so the total number of points covered by Baer subspaces is $\left(q^{2n-2}+q^{2n-4}+\cdots+q^2+1\right)\left(q^{2n}-1\right)$. Hence, the number of Baer subspaces is exactly $(q-1)(q^{2n-2}+q^{2n-4}+\cdots+q^2+1)$. ∎

From this point on we will restrict ourselves to the case when $n=2$. We will use $\mathcal{P}_0$ to denote the mixed partition given above in this case.

3. PSEUDO-REGULI AND PARTITIONS FROM $\mathcal{P}_0$

Let l_0 and l_∞ be the two distinct lines of the partition $\mathcal{P}_0$. Recall that the partition $\mathcal{P}_0$ is made up of orbits of a group Θ of order q^3+q^2+q+1 that acts as a Singer group on the points of the Baer subspaces of $\mathcal{P}_0$. Let σ be a generator of Θ.

LEMMA 3.1. *If a line* l *of* Π *meets each of the lines* l_0 *and* l_∞ *in a point, then* l *meets* $q-1$ *of the Baer subspaces of* $\mathcal{P}_0$ *in a Baer subline and is disjoint from the remaining* $q^2(q-1)$ *Baer subspaces.*

Proof. Let l be a line of Π meeting each of l_0 and l_∞ in a point, say R_1 and R_2 respectively. Furthermore, suppose that l meets a Baer subspace B_0 of $\mathcal{P}_0$ in a unique point Q. Because of the structure of the group Θ, we know that $R_i^{\sigma^{q^2+1}} = R_i$ for each i. Hence $l^{\sigma^{q^2+1}} = l$, which implies that $Q^{\sigma^{q^2+1}} = Q$. But this contradicts the action of Θ on the points of B_0. ∎

This small result leads to the first new type of partition. Let l be any line of Π that meets each of l_0 and l_∞ in a point and consider the orbit of l under the cyclic group $H = \langle \sigma^{q+1} \rangle$ of order $q^2 + 1$. The lines in this orbit each meet l_0 and l_∞ in a single point, and it is not hard to show that these lines induce a regular spread in $q - 1$ of the Baer subspaces of $\mathcal{P}_0$. Replacing these Baer subspaces and the two lines l_0 and l_∞ with the lines of l^H, we generate a partition of Π containing $q^2 + 1$ lines and $q^2(q - 1)$ Baer subspaces. We will call this new partition $\mathcal{P}_0'$.

We should note that the lines of l^H form a *pseudo-regulus*, which was originally defined in [5].

Definition 3.1. Given a regular spread $\mathcal{S}$ of $\Pi_0 \cong \mathrm{PG}(3, q)$ embedded in $\Pi = \mathrm{PG}(3, q^2)$, let $\mathcal{F}$ be the partial spread of Π obtained by extending the lines of $\mathcal{S}$ to the space Π. This partial spread $\mathcal{F}$ is called a **pseudo-regulus** of Π.

Theorem 3.1. *(Freeman, [5]) If $\mathcal{F}$ is a pseudo-regulus of* $\mathrm{PG}(3, q^2)$, *then $\mathcal{F}$ is contained in a spread of* $\mathrm{PG}(3, q^2)$.

With this result, other mixed partitions arise. It is shown in [5] that the point set covered by the lines of a pseudo-regulus can always be partitioned into $q - 1$ disjoint (transversal) Baer subspaces and 2 transversal lines. Hence, by taking any spread containing a pseudo-regulus, one can replace the pseudo-regulus with the $q - 1$ transversal Baer subspaces and 2 transversal lines, yielding a mixed partition with $q^4 - q^2 + 2$ lines and $q - 1$ Baer subspaces.

Lemma 3.2. *If a line l of Π meets exactly one of the lines l_0 and l_∞ in a point, then l meets exactly q^2 of the Baer subspaces of $\mathcal{P}_0$ in a single point and is disjoint from the remaining $(q - 1)(q^2 + 1) - q^2$ Baer subspaces.*

Proof. Without loss of generality, let l be a line of Π meeting l_0 in a point, say R, with $l \cap l_\infty = \emptyset$. Also, for contradiction, suppose that l meets a Baer subspace B of $\mathcal{P}_0$ in a Baer subline $\bar{l}$. Then, as before, $R^{\sigma^{q^2+1}} = R$, which means that the lines l and $l^{\sigma^{q^2+1}}$ share at least one common point. If $l \neq l^{\sigma^{q^2+1}}$, then $\bar{l}$ and $\bar{l}^{\sigma^{q^2+1}}$ are coplanar and, since they are both contained in B, they must intersect in a point of B. Hence, we have two distinct lines sharing two common points, a contradiction. Therefore, $\bar{l} = \bar{l}^{\sigma^{q^2+1}}$ which implies $l = l^{\sigma^{q^2+1}}$. The orbit of l under Θ could not be of any shorter length because of the action of Θ on the points of l_0. Therefore, the orbit l^Θ contains exactly $q^2 + 1$ lines.

The points of any Baer subspace in $\mathcal{P}_0$ form a Θ-orbit of length $q^3 + q^2 + q + 1$. This means that l could not possibly meet a Baer subspace in a single point. Thus, l meets every Baer subspace of $\mathcal{P}_0$ in 0 or $q+1$ points. Therefore, $q^2 + 1 = 1 + k(q + 1)$ where k is the number of Baer subspaces that meet l in a Baer subline. This implies $(q + 1)|q^2$, a contradiction. ▮

Lemma 3.3. *If a line l of Π is disjoint from both l_0 and l_∞, then l meets at most one of the Baer subspaces of $\mathcal{P}_0$ in a Baer subline.*

Proof. We prove the contrapositive. Let l be a line of Π and suppose that l meets two distinct Baer subspaces of $\mathcal{P}_0$, say B_m and B_n, in Baer sublines m and n, respectively. Now consider the orbit of m under the group Θ. Either m^Θ is a full orbit of length $q^3 + q^2 + q + 1$, or m^Θ forms a regular spread of B_m (see [3]). If m^Θ is a full orbit, then there are two distinct lines of m^Θ that intersect, forcing the corresponding Baer sublines of B_n to be coplanar and, therefore, to intersect in a point of B_n. Since they have two points in common, this implies that the two lines are the same, contradicting their distinctness.

Hence, m^Θ must be a line-orbit of length $q^2 + 1$. Now suppose that l meets another Baer subspace different from B_m and B_n in a unique point R. Then R^Θ is a point orbit of length $q^2 + 1$, a contradiction. Hence, l must meet all Baer subspaces in 0 or $q + 1$ points. By a simple counting argument, this implies that l meets each of l_0 and l_∞ in a unique point. ▮

Putting all of the above lemmas together, we find that there are exactly four different intersection patterns of lines of Π (different from l_0 and l_∞) with the partition $\mathcal{P}_0$.

Type 1: lines that meet both l_0 and l_∞ in a point, meet exactly $q - 1$ of the Baer subspaces in a Baer subline, and are disjoint from the remaining $q^2(q - 1)$ Baer subspaces

Type 2: lines that meet exactly one of l_0 and l_∞ in a unique point, meet exactly q^2 Baer subspaces in a unique point, and are disjoint from the remaining $(q - 1)(q^2 + 1) - q^2$ Baer subspaces

Type 3: lines skew to both l_0 and l_∞ that meet exactly one Baer subspace in a Baer subline, exactly $q^2 - q$ Baer subspaces in a unique point, and are disjoint from the remaining $q^3 - 2q^2 + 2q - 2$ Baer subspaces

Type 4: lines skew to both l_0 and l_∞ that meet exactly $q^2 + 1$ Baer subspaces in a unique point, and are disjoint from the remaining $(q - 2)(q^2 + 1)$ Baer subspaces

TABLE 1.

Number of lines of different types

Line Type	Number
Type 1	$(q^2+1)^2$
Type 2	$2(q^2+1)^2(q^2-1)$
Type 3	$q(q^2+1)^2(q^2-1)$
Type 4	$q(q-1)(q^3-q^2-q-1)(q^3+q^2+q+1)$

For the purposes of this section, we are particularly interested in the Type 1 and Type 4 lines. Each of these will give rise to a mixed partition. We start by counting the number of lines of each type. There are clearly $(q^2+1)^2$ Type 1 lines. The lines of Type 1 or Type 2, along with l_0 and l_∞, can be counted by inclusion/exclusion giving us $2\left[(q^2+1)(q^4+q^2)+1\right]-(q^2+1)^2 = 2q^6+3q^4+1$. Hence there are $2(q^2+1)^2(q^2-1)$ Type 2 lines. The total number of Baer sublines of Π contained in a Baer subspace of $\mathcal{P}_0$ is $(q^3-1)(q^2+1)^2$, but $(q^2+1)^2(q-1)$ of these Baer sublines generate Type 1 lines. Hence, the total number of Type 3 lines is $q(q^2+1)^2(q^2-1)$. We can now count the number of Type 4 lines by subtracting the total number of Type 1, 2, and 3 lines (plus an extra 2 for l_0 and l_∞) from the total number of lines of Π, $(q^4+1)(q^4+q^2+1)$. This gives us exactly $q(q-1)(q^3-q^2-q-1)(q^3+q^2+q+1)$ Type 4 lines. A summary of the number of lines of the different types appears in Table 1.

THEOREM 3.2. *There exists a mixed partition $\mathcal{P}_0''$ of Π with q^3+q^2+q+3 lines and $(q-2)(q^2+1)$ Baer subspaces.*

Proof. We construct such a mixed partition from the partition $\mathcal{P}_0$. From the discussion above, one can always find a (Type 4) line l of Π that meets exactly q^2+1 Baer subspaces of $\mathcal{P}_0$ in a unique point. By starting with $\mathcal{P}_0$ and replacing these q^2+1 Baer subspaces with the set of lines in l^Θ, we get the desired mixed partition. ∎

In summary, Type 1 lines give us the partition $\mathcal{P}_0'$ and Type 4 lines give us the partition $\mathcal{P}_0''$.

4. A PARTITION FROM A REGULAR SPREAD

The objective of this section is to create a new type of mixed partition using a special group action. For the following, we will be working in $\Pi = \mathrm{PG}(3, q^2)$ where **we now assume q is odd**. Let $K = GF(q^2)$ with primitive element β. We let F be the subfield $GF(q)$ of K, so that $\omega = \beta^{q+1}$ is a primitive element of F. We will also make use of the special element $\epsilon = \beta^{\frac{q+1}{2}}$, where one can easily show that $\epsilon^q = -\epsilon$.

For our construction, we will take a regular spread in Π and find a Baer subspace that meets each line of the regular spread in at most one point. Note that the only possible intersection sizes are $0, 1$, or $q + 1$ where the $q+1$ intersection size corresponds to a Baer subline. So, we are looking for a Baer subspace that does not meet any of the lines of the regular spread in a Baer subline.

We start with a representation of a regular spread. In Bruck [4] it is shown that the lines

$$\left\{ l_{(x,y)} = \langle (x, y, 1, 0), (\beta y, x, 0, 1) \rangle : x, y \in K \right\}$$

together with the extra line

$$l_\infty = \langle (1, 0, 0, 0), (0, 1, 0, 0) \rangle$$

form a regular spread of Π. These lines are constructed from the ruling families of certain quadrics, and the coordinates given here are carefully determined in [2]. We construct our desired regular spread from this model where the basis for the underlying vector space is non-standard. Alternatively, we can think of the change of basis as the application of some collineation.

Let

$$M_\phi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\epsilon & 0 \\ 0 & 1 & \epsilon & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix M_ϕ induces a collineation ϕ on Π. Applying this collineation to Π, we can write the images of the generators for the lines of our new regular spread as $\langle (x, y + 1, -\epsilon(y - 1), 0) \rangle$ and $\langle (\beta y, x, -\epsilon x, 1) \rangle$. Let $\mathcal{S}^*$ be the spread obtained from these lines and let B_0 be the natural Baer subspace of Π, that is, the one whose homogeneous coordinates are in the subfield $GF(q)$.

THEOREM 4.1. *Every line of $\mathcal{S}^*$ meets B_0 in at most one point.*

Proof. Consider the line $l^{\phi}_{(x,y)}$ of $\mathcal{S}^*$. An arbitrary point on this line can be written as either

$$\langle(\beta y+\lambda x, x+\lambda(y+1), -\epsilon x-\epsilon\lambda(y-1), 1)\rangle$$

for some $\lambda \in K$, or

$$\langle(x, y+1, -\epsilon(y-1), 0)\rangle.$$

Hence, each line of the spread has just one point whose last coordinate is zero. So, if a line of $\mathcal{S}^*$ meets B_0 in a Baer subline, then that Baer subline would have at least two points whose last coordinates are non-zero. For contradiction, suppose the following two vectors induce points of $l^{\phi}_{(x,y)}$ that are also both in B_0:

$$\mathbf{q}_1 = (\beta y+\lambda_1 x, x+\lambda_1(y+1), -\epsilon x-\epsilon\lambda_1(y-1), 1)$$

and

$$\mathbf{q}_2 = (\beta y+\lambda_2 x, x+\lambda_2(y+1), -\epsilon x-\epsilon\lambda_2(y-1), 1),$$

where $\lambda_1 \neq \lambda_2$. We will start by looking at 2 special cases.

First, suppose that $y=1$. Now, $\mathbf{q}_1-\mathbf{q}_2$ must also induce a point of B_0 and, moreover, since $\mathbf{q}_1$ and $\mathbf{q}_2$ both have all of their coordinates in F, $\mathbf{q}_1-\mathbf{q}_2$ must have all of its coordinates in F as well. But the second coordinate of $\mathbf{q}_1-\mathbf{q}_2$ implies $\lambda_1-\lambda_2 \in F$. Since $\lambda_1-\lambda_2 \neq 0$, we get $x \in F$ from the first coordinate. But from the third coordinate of $\mathbf{q}_1$, $\epsilon x \in F$. The only possibility is that $x=0$. If $x=0$, the first coordinate of $\mathbf{q}_1$ is β which is clearly not in F, a contradiction.

A similar argument leads to a contradiction in the case when $y=-1$. Hence, we can assume from this point on that $y \neq 1$ and $y \neq -1$.

Going back to our original forms for $\mathbf{q}_1$ and $\mathbf{q}_2$, we get that

$$\mathbf{q}_1-\mathbf{q}_2 = (x(\lambda_1-\lambda_2), (y+1)(\lambda_1-\lambda_2), -\epsilon(y-1)(\lambda_1-\lambda_2), 0)$$

also induces a point of B_0. Since $y \neq 1$, we can right normalize this vector and rewrite the homogeneous coordinates for the associated point as

$$\left(\frac{x}{-\epsilon(y-1)}, \frac{y+1}{-\epsilon(y-1)}, 1, 0\right).$$

So, $\frac{x}{-\epsilon(y-1)} \in F$ and $\frac{y+1}{-\epsilon(y-1)} \in F$. In particular, note that $y \neq 0$, and the Frobenius map acts as the identity on these values. Hence,

$$\frac{y^q+1}{\epsilon(y^q-1)} = \frac{y+1}{-\epsilon(y-1)}.$$

Cross multiplying and cancelling gives us $y^q = \frac{1}{y}$. By a similar argument applied to $\frac{x}{-\epsilon(y-1)}$, we obtain $x^q y = x$.

Since, $\mathbf{q}_1$ and $\mathbf{q}_2$ are both normalized, $x + \lambda_i(y+1) \in F$ for $i = 1, 2$. Hence, the vector $[x+\lambda_2(y+1)]\mathbf{q}_1 - [x+\lambda_1(y+1)]\mathbf{q}_2$ has all of its coordinates in F. Now, since $y \neq -1$, we can right normalize this vector to obtain

$$\mathbf{q} = \left(\beta y + \frac{-x^2}{y+1}, 0, \frac{\epsilon x(y-1)}{y+1} - \epsilon x, 1\right).$$

Since $\mathbf{q}$ induces a vector in B_0, $\epsilon\left(\frac{x(y-1)}{y+1} - x\right) \in F$. Again using the Frobenius map, together with the two identities obtained earlier, we obtain $2x = -2x$. Since q is odd, this implies $x = 0$.

Now, since $x = 0$, we can rewrite $\mathbf{q}_1$ as $(\beta y, \lambda_1(y+1), -\epsilon\lambda_1(y-1), 1)$ which implies that $\beta y \in F$. But $\beta y \in F \Rightarrow \frac{\beta^q}{y} = \beta y \Rightarrow y^2 = \beta^{q-1}$. So $y = \beta^{\frac{q-1}{2}}$ which implies

$$1 = y^{q+1} = \beta^{\frac{q^2-1}{2}} = -1,$$

a contradiction. Hence, in all cases we get a contradiction if we assume that two points of a line of $\mathcal{S}^*$ lie in B_0. ▮

We now carefully explain how to use this Baer subspace to generate a new type of mixed partition of Π.

THEOREM 4.2. *In $\Pi = \mathrm{PG}(3, q^2)$, q odd, there exists a mixed partition $\mathcal{P}_S$ with q^2+1 Baer subspaces and $q^4 - q^3 - q^2 - q$ lines.*

Proof. We start with the regular spread $\mathcal{S}^*$ described above. By the previous theorem, we know that the Baer subspace B_0 meets each line of $\mathcal{S}^*$ in at most one point. Every regular spread has an associated Bruck Kernel [4], a cyclic group of order q^2+1 that acts regularly on the points of each line of the regular spread. Let ξ denote the Bruck Kernel associated with $\mathcal{S}^*$. By applying ξ to B_0, we get an orbit of q^2+1 Baer subspaces, pairwise disjoint by the regularity, that cover the point set determined by the lines of $\mathcal{S}^*$ intersecting B_0 in exactly one point. These q^2+1 Baer subspaces together with the lines of $\mathcal{S}^*$ that do not intersect B_0 form the desired mixed partition that we call $\mathcal{P}_S$. ▮

5. THE EXISTENCE OF A *REGULUS TYPE* MIXED PARTITION

We can give another example of a mixed partition with interesting structure. It was discovered in [11] that there is a mixed partition of PG(3, 4) whose lines form a regulus. It was proven in [9] that this partition is part of an infinite family of mixed partitions for q even. We reference [9] for the complete proof of the existence of this partition, but give an overview of the proof here.

We define the point set covered by the lines of our partition. Since these lines are supposed to form a regulus, their point set is a hyperbolic quadric. Let

$$\mathcal{Q} = \{\langle(x_0, x_1, x_2, x_3)\rangle : x_0x_2 - x_1x_3 = 0\}.$$

The quadric $\mathcal{Q}$ is certainly a non-degenerate hyperbolic quadric and so contains two ruling families of lines.

Now consider the set of matrices

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & b & 0 \\ a & 0 & 0 & b \end{bmatrix} : a \in K, b \in F^* \right\},$$

where F^* is the set of non-zero elements in the finite field F. One can easily show that these matrices induce a collineation group G on Π that fixes the hyperbolic quadric $\mathcal{Q}$. Moreover, there are exactly $q^3 + q^2 + q + 1$ point orbits formed on the points off $\mathcal{Q}$. It can be shown that, for q even, one can always find a Baer subspace B that meets each of these orbits in exactly one point. To construct such a Baer subspace, choose a primitive element β of K such that $\mathrm{Tr}_{K/F}(\beta) = 1$. Then, letting $\alpha = \beta^{q-1}$, one can show that the points generated by the following vectors span the desired Baer subspace B: $(\alpha^4, \alpha^3, 1, 0), (\alpha, \alpha^2, 0, 1), (\alpha^3, \alpha^2, 1, 0), (1, \alpha, 0, 1)$, and $\left(\frac{\alpha^3+1}{\alpha}, \frac{1}{\beta}, 1, 1\right)$. By taking the Baer subspaces in the orbit B^G together with either ruling family of lines for $\mathcal{Q}$, we can construct a mixed partition containing $q^2 + 1$ lines and $q^2(q - 1)$ lines (see [9] for more details).

6. CONCLUSION

It is well known that there is a close relationship between finite affine translation planes and $(n - 1)$-spreads of $\mathrm{PG}(2n - 1, q)$. Moreover, it is shown in [6] that any mixed partition of $\mathrm{PG}(2n - 1, q^2)$ can be used to construct a $(2n-1)$-spread of $\mathrm{PG}(4n-1, q)$. Hence, it is natural to look for more examples of such mixed partitions to help in the search for potentially new translation planes.

We now see that there are many examples of mixed partitions. The idea of several mixed partitions generating equivalent spreads is explored in [8] and [10]. It would be interesting to find properties of a mixed partitions $\mathcal{P}$ which would help in determining the type of translation plane constructed from $\mathcal{P}$. This could possibly help in the classification of certain types of translation planes, or at least provide new models for existing planes.

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