

Combinatorial Designs and Scheduling Golf

Monika I. Rut

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During the summer of 2002, I spent time working on a seemingly simple problem. We have a group of twelve friends who want to spend a summer playing golf. They would like to get together three times a week in three groups of four. Is it possible to arrange our golfers so that everybody plays with everybody else in some regular way?

Simple counting tells us that the total number of groups of four is $\binom{12}{4} = 495$. Moreover, the total number of ways to group 12 golfers into 3 groups of four is given by the multinomial coefficient $\binom{12}{4\ 4\ 4} = 34650$. This certainly is not a feasible number for the summer fun, not even a whole decade. As simple as it sounds, it gets more complicated when we start to think about it in a more organized manner. Let's say that we would like to make sure that in our schedule every golfer plays with every other golfer exactly the same number of times. In order to cover up a whole summer we would need to create about 35 to 50 such days where 3 groups of 4 golfers meet. Last but not least it would be nice to make every group of 3 golfers meet at least once. Why? Let's say that John plays with Tom and on some other day John plays with Monika. By ensuring that every golfer plays with every other golfer we could be certain that on some day Tom will also play with Monika; however, if we do not ensure that every triple plays together at least once then Tom, John and Monika may never meet together.

A quiet nice solution comes up from combinatorial designs. In particular the balanced incomplete block design, or BIBD for short, can be very helpful in our problem.

Definition 1 *A design is a set of v points together with a collection of subsets of size k of these points, called blocks, with the property that any t points lie in exactly λ blocks. Such a design with these parameters is called a $t - (v, k, \lambda)$ design.*

As an example, suppose we want every pair of golfers to play together exactly once. Then, the number of golfers $v = 12$, they play in groups of $k = 4$ and every $t = 2$ golfers play together exactly $\lambda = 1$ times. So we seek a 2-(12, 4, 1) design. Does such design exist?

Proposition 1 [1] *In a BIBD each points is contained in*

$$r = \frac{\lambda(v - 1)}{k - 1}$$

blocks.

When using our example, each golfer plays in a certain number of groups of 4 where each group corresponds to a block. Here, r is the number of days that each golfer gets to play. With $\lambda = 1$ our value for r is $\frac{11}{3}$ which is not an integer. Hence, this design does not exist. When we make $\lambda = 2$, r becomes $\frac{22}{3}$. However, making $\lambda = 3$ gives an integer value for r , namely $r = 11$. So the 2-(12, 4, 3) design might provide a good solution to our problem. This means that 12 players will play for 11 days in 3 groups of 4 each day and every 2 golfers will play with one another exactly 3 times.

We start by numbering our players from 0 to 11. Since creation of our design will involve modular arithmetic, it is the best to start from 0. Also, we rename player 11 with the symbol ∞ . Fortunately the above design does exist and it can be created from the following base blocks [3].

$$\{\infty, 3, 7, 9\}, \{0, 1, 4, 10\}, \{2, 5, 6, 8\}$$

This design is called 1-rotational since it is obtained through mapping the base blocks from i to $i + 1 \pmod{11}$ while fixing ∞ . After creating all 11 days we can just substitute 11 for ∞ . The second day for instance would consist of the three blocks

$$\{\infty, 4, 8, 10\}, \{0, 1, 2, 5\}, \{3, 6, 7, 9\}$$

and so on.

Now we have a schedule for 11 days, but it only solves our problem for about 4 weeks and the summer is not yet over. We could repeat the same design once the eleventh day is over. By doing so we will have the same groups of golfers play together again and such a solution will not decrease the numbers of triples that never meet. In the above design, 99 groups of 3 players never meet. We used the software package Magma [2] to find such triples.

We consider the possibility of mapping additive inverses modulo 11 in order to obtain 11 more days to add to our schedule. In other words, player 1 changes her or his place with player 10, 2 with 9, 3 with 8, 4 with 7, 5 with 6 and 0 with 11. We will call this new design the *inverse design*. For example, day one in the inverse design contains the blocks

$$\{0, 2, 4, 8\}, \{1, 7, 10, 11\}, \{3, 5, 6, 9\}.$$

Now we have 22 days covered, but unfortunately this still does not cover the whole summer.

We already know that making every possible group of four golfers play together is impossible to obtain in a summer. Hence, we concentrate on minimizing the groups of three golfers which never meet. By simple counting there are $\binom{12}{3} = 220$ possible triples.

Again with help from Magma we can find out how many such triples appear in both the original and inverse designs. Magma shows us that 40 triples never appear together. By creating an inverse design we were able to reduce those triples by 59 and increase our scheduled days by 11.

Can we do better? Can we cover more days? We can start by checking if there exists a number that added to each variety in our original design (mod 12) would create more non-repeating blocks. Let us call triples that never appear *bad triples*. We can take our collection of bad triples and find out if there exists a number from 0 to 11 that added to each point in every triple would create a new triple which does not appear in our original design. Fortunately with the aid of a simple algorithm and the quickness of Magma we were able to conclude that 6 can bring the number of bad triples to zero. For instance, if we add 6 to every number in day one of the original design we obtain

$$\{1, 3, 5, 9\}, \{4, 6, 7, 10\}, \{0, 2, 8, 11\}.$$

In the same fashion we can apply this technique to the inverse design. At the end we are left with four designs which we concatenate to form our final solution.

Original Design			Original Design + 6		
{0, 1, 4, 10}	{2, 5, 6, 8}	{3, 7, 9, 11}	{6, 7, 10, 4}	{8, 11, 0, 2}	{9, 1, 3, 5}
{0, 1, 2, 5}	{3, 6, 7, 9}	{4, 8, 10, 11}	{6, 7, 8, 11}	{9, 0, 1, 3}	{10, 2, 4, 5}
{0, 5, 9, 11}	{1, 2, 3, 6}	{4, 7, 8, 10}	{6, 11, 3, 5}	{7, 8, 9, 0}	{10, 1, 2, 4}
{0, 5, 8, 9}	{1, 6, 10, 11}	{2, 3, 4, 7}	{6, 11, 2, 3}	{7, 0, 4, 5}	{8, 9, 10, 1}
{0, 2, 7, 11}	{3, 4, 5, 8}	{1, 6, 9, 10}	{6, 8, 1, 5}	{9, 10, 11, 2}	{7, 0, 3, 4}
{0, 2, 7, 10}	{4, 5, 6, 9}	{1, 3, 8, 11}	{6, 8, 1, 4}	{10, 11, 0, 3}	{7, 9, 2, 5}
{0, 1, 3, 8}	{2, 4, 9, 11}	{5, 6, 7, 10}	{6, 7, 9, 2}	{8, 10, 3, 5}	{11, 0, 1, 4}
{0, 6, 7, 8}	{1, 2, 4, 9}	{3, 5, 10, 11}	{6, 0, 1, 2}	{7, 8, 10, 3}	{9, 11, 4, 5}
{0, 4, 6, 11}	{1, 7, 8, 9}	{2, 3, 5, 10}	{6, 10, 0, 5}	{7, 1, 2, 3}	{8, 9, 11, 4}
{0, 3, 4, 6}	{2, 8, 9, 10}	{1, 5, 7, 11}	{6, 9, 10, 0}	{8, 2, 3, 4}	{7, 11, 1, 5}
{0, 3, 9, 10}	{2, 6, 8, 11}	{1, 4, 5, 7}	{6, 9, 3, 4}	{8, 0, 2, 5}	{7, 10, 11, 1}
Inverse Design			Inverse Design + 6		
{0, 2, 4, 8}	{3, 5, 6, 9}	{1, 7, 10, 11}	{6, 8, 10, 2}	{9, 11, 0, 3}	{7, 1, 4, 5}
{0, 1, 3, 7}	{2, 4, 5, 8}	{6, 9, 10, 11}	{6, 7, 9, 1}	{8, 10, 11, 2}	{0, 3, 4, 5}
{0, 2, 6, 11}	{1, 3, 4, 7}	{5, 8, 9, 10}	{6, 8, 0, 5}	{7, 9, 10, 1}	{11, 2, 3, 4}
{0, 1, 5, 10}	{2, 3, 6, 11}	{4, 7, 8, 9}	{6, 7, 11, 4}	{8, 9, 0, 5}	{10, 1, 2, 3}
{0, 4, 9, 11}	{1, 2, 5, 10}	{3, 6, 7, 8}	{6, 10, 3, 5}	{7, 8, 11, 4}	{9, 0, 1, 2}
{0, 3, 8, 10}	{1, 4, 9, 11}	{2, 5, 6, 7}	{6, 9, 2, 4}	{7, 10, 3, 5}	{8, 11, 0, 1}
{0, 2, 7, 9}	{1, 4, 5, 6}	{3, 8, 10, 11}	{6, 8, 1, 3}	{7, 10, 11, 0}	{9, 2, 4, 5}
{0, 1, 6, 8}	{3, 4, 5, 11}	{2, 7, 9, 10}	{6, 7, 0, 2}	{9, 10, 11, 5}	{8, 1, 3, 4}
{0, 5, 7, 11}	{2, 3, 4, 10}	{1, 6, 8, 9}	{6, 11, 1, 5}	{8, 9, 10, 4}	{7, 0, 2, 3}
{0, 4, 6, 10}	{1, 2, 3, 9}	{5, 7, 8, 11}	{6, 10, 0, 4}	{7, 8, 9, 3}	{11, 1, 2, 5}
{0, 3, 5, 9}	{4, 6, 7, 10}	{1, 2, 8, 11}	{6, 9, 11, 3}	{10, 0, 1, 4}	{7, 8, 2, 5}

By creating such designs we were able to minimize the number of bad triples to zero. Hence, all possible groups of three golfers play together at least once. Every two golfers play together exactly 3 times in each design (12 times total) and we were able to reduce a number of repeating groups of 4 to only eight. Is this the best we can do? We were able to cover 44 days, every two golfers play together exactly 12 times and every possible triple plays together at least once. In my opinion this seems to be a rather good and systematic approach and most important it will keep our 12 golfers busy for 44 days without making them repeat their partners too often.

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References

- [1] R. A. Brualdi, *Introductory Combinatorics*, third edition, Prentice Hall, Upper Saddle River, NJ, 1999.
- [2] J. Cannon and C. Playoust, *An Introduction to MAGMA*, Univ. of Sydney, Sydney, Australia, 1993.
- [3] C.J. Colbourn and J.H.Dinitz, editors, *CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton, FL, 1996.