An Elementary Solution to the Ménage Problem

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1 Introduction

The ménage problem asks for the number of ways to seat n husbands and n wives at a circular table with alternating men and women so that no one is seated next to their spouse. This problem was first labeled as the *probleme des ménages* by Edouard Lucas in 1891 [2]. The probleme des ménages is a natural extension of another problem called the probleme des rencontres which asks for the number of permutations of $\{1, 2, ..., n\}$ in which every integer is out of place [4] (also known as a *derangement*). Most of the solutions tell how to compute a recurrence relation or a generating function instead of giving an explicit formula. Touchard was the first person to give an explicit formula although he did not give a proof [2].

When $n = 3$, the problem is easy to visualize. There are, in fact, only two such arrangements and they are displayed in Figure 1. Note that, in our interpretation, we avoid thinking of cyclic permutations as different. In other words, if everybody stands up and moves over one seat, we consider this an equivalent arrangement.

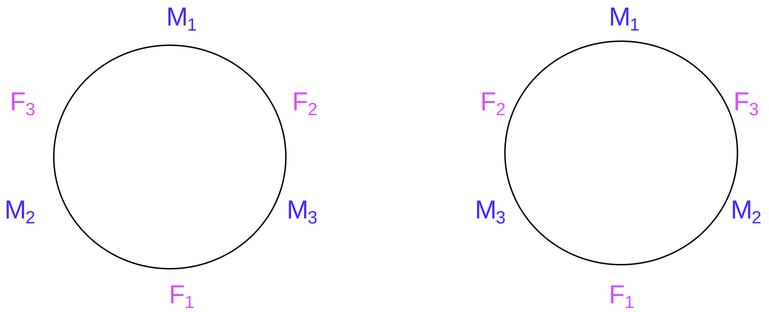

Figure 1: Configurations with 3 couples

2 A combinatorial solution

We provide a solution to the probleme des ménages using only elementary counting techniques. The first lemma here can be thought of as a warm up to the second lemma which will actually be required in our solution. Recall that the number of ways to choose k objects from n objects where order does not matter and repeats are allowed is $\binom{n+k-1}{k}$. Also, by deciding which elements to choose or avoid, we obtain the elementary identity $\binom{n}{k} = \binom{n}{n-k}$.

Lemma 2.1 *The number of ways of selecting k objects, no two consecutive, from n objects arrayed in a row is $\binom{n-k+1}{k}$.*

Proof: We know that every time we select our k objects, we will also have to choose $k-1$ objects, each of which will go between an adjacent pair of the k selected objects. So there are $n-(k+k-1) = n-2k-1$ objects left and we must decide where to put them. These objects can be in any of the $k+1$ spaces, either in front of the first object chosen, after the k^{th} object chosen, or in between any two of the k chosen objects. For these $n-2k-1$ objects we could choose an available space more than once and certainly the order of selection is irrelevant. Referring to the notation above, our "n"$= k+1$, and our "k" $= n-2k+1$. Thus, our count is $\binom{(k+1)+(n-2k+1)-1}{n-2k+1} = \binom{n-k+1}{n-2k+1}$. Now using the last identity above, $\binom{n-k+1}{n-2k+1} = \binom{n-k+1}{n-k+1-(n-2k+1)} = \binom{n-k+1}{k}$. ■

Now we look at the lemma we need to solve the ménage problem. For the moment, we will assume that the chairs around our table are numbered. Hence, cyclic shifts of the people at our table will be considered different. We will then adjust our solution at the end to eliminate these equivalent arrangements.

Lemma 2.2 *The number of ways of selecting k objects, no two consecutive, from n objects arrayed in a circle is* $\binom{n-k}{k}\frac{n}{(n-k)}$.

Proof: There are n choices for the first object selected. We now have to select the remaining $k-1$ objects and the k objects that are between them, so they will not be consecutive. This leaves $n-1-(k-1+k) = n-2k$ objects unaccounted for and there are k positions these objects could be in. Hence, as before, we have $\binom{k+n-2k-1}{n-2k} = \binom{n-k-1}{n-2k} = \binom{n-k-1}{k-1}$ ways to select these $n-2k$ positions. We must now multiply this by the n choices for the location of the first object and divide by k since we could pick the same group again by starting with a different member, and this can happen in k ways. It can easily be shown that $\binom{n-k-1}{k-1}\frac{n}{k} = \binom{n-k}{k}\frac{n}{(n-k)}$. ■

We are now ready to move on to a solution of the ménage problem. Earlier solutions to the ménage problem do not follow the general way we think of arranging people around a table. Usually if people are seated in a certain order around a circular table and they all decide to get up and shift left or right a fixed number of seats, they are considered to be in the same "arrangement." However, earlier solutions to the ménage problem count these as two different arrangements. If we were to count these as equivalent then there would be only two solutions to the problem when $n = 3$ as pictured earlier. Also, when $n = 4$ there are 96 arrangements, but, as stated before, this over-counts. There are actually only 12 arrangements up to cyclic equivalence.

For the final solution we will need to define u_n as the number of permutations of $\{1, 2, ..., n\}$ that do not satisfy any of the following conditions:

1. person 1 is in the $1st$ position
2. person 1 is in the $2nd$ position
3. person 2 is in the $2nd$ position
4. person 2 is in the $3rd$ position

$\vdots$

2n-1. person n is in the nth position

2n. person n is in the $1st$ position.

Theorem 2.3 *The number of ways to seat n couples, alternating men and women, so that no one is sitting next to their spouse is* $2n!u_n$.

Proof: Let us start by seating the men. They can occupy either the odd or the even numbered seats and they can be arranged in $n!$ ways. The women can now be seated in u_n ways, so as to avoid sitting next to their spouses. So we have a solution to the problem if we can determine a value for u_n.

In order to find u_n, we appeal to the principle of inclusion/exclusion. The number of ways to arrange the women without any stipulations is $n!$. From this we need to remove all the permutations that meet any of the conditions listed above. Let A_i be the set of all permutations satisfying condition i given above. Then using inclusion/exclusion

$$\begin{aligned} u_n &= n! - |A_1 \cup A_2 \cup A_3 \cup ... \cup A_{2n}| \\ &= n! - [|A_1| + |A_2| + ... + |A_{2n}| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| - ... \\ &\quad +|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \cdots] \end{aligned}$$

This expression can be reduced considerably. For instance, the cardinality of the set $A_1 \cap A_2$ is zero since woman number 1 cannot be in both the first and second seat. In general, if any two consecutive conditions are chosen the

n	2	3	4	5	6	7
$(n-1)!u_n$	0	2	12	312	9600	416880

Table 1: Solutions to the ménage problem

cardinality of the corresponding set will be zero. So, we need to know the number of sets that do not have cardinality zero. We can do this by using Lemma 2.2. The number of ways to pick two non-consecutive conditions from the $2n$ conditions is $\binom{2n-2}{2}\frac{2n}{(2n-2)}$. Also, the cardinality of each of these sets is $(n-2)!$ since two conditions are now fixed. So, two of the women have already been seated and the other women can be seated in $(n-2)!$ ways. Thus, the number of ways in which any two conditions can be satisfied at the same time is $\frac{2n}{2n-2}\binom{2n-2}{2}(n-2)!$. The same is true for the case where we are trying to choose three pairwise nonconsecutive conditions. From the lemma we know that the number of ways to do this is $\binom{2n-3}{3}\frac{2n}{(2n-3)}$, and the number of ways to arrange the remaining women is $(n-3)!$. This can be extended naturally. Hence, $u_n = n! - \frac{2n}{2n-1}\binom{2n-1}{1}(n-1)! + \frac{2n}{2n-2}\binom{2n-2}{2}(n-2)! - \frac{2n}{2n-3}\binom{2n-3}{3}(n-3)! + \cdots$ Therefore, the solution to the menage problem can be written as

$$2n!\sum_{i=0}^{n}(-1)^i\frac{2n}{2n-i}\binom{2n-i}{i}(n-i)!.$$

However, as stated above, this expression over counts. To eliminate cyclic equivalence, we must divide by $2n$ which results in

$$(n-1)!\sum_{i=0}^{n}(-1)^i\frac{2n}{2n-i}\binom{2n-i}{i}(n-i)!.$$

As the number of couples increases, the solution to the ménage problem grows rapidly. Table 1 show the solution to the ménage problem for small n.

3 Conclusion

There are many other problems similar to that of the ménage problem. For instance, if two people turn up cards alternatively from two different shuffled packs of cards, what is the probability that before the decks run out two identical cards are showing at the same time [2]? Another problem similar to the menage problem involves a square table and the stipulation that no one can sit next to or across from his or her partner [1].

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References

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