Investigation of a Min-Max Scheduling Problem

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This thesis by **Ryan Johnson** is accepted in its present form as satisfying the thesis requirement for Honors in Mathematics.

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Abstract

We investigate a min-max problem in the theory of combinatorial designs. The problem involves an attempt to partition students into groups of size k over several days, but subject to the condition that no pair of students works together more than once. A list of partitions satisfying the condition and spanning i days is called a *schedule* of length i. A schedule that cannot be extended is called *maximal*. We wish to determine the minimum length of a maximal schedule. Through several proofs and creative methods, we are able to determine this minimum number in the case when $k = 2$ (and the students are partitioned into pairs). This allows us to look forward to generalizations with $k \geq 3$.

1 Introduction and Preliminaries

We wish to examine a scheduling problem involving combinatorial designs. A *combinatorial design* is a collection of v points, together with a collection of subsets of these points, called *blocks*, that satisfy a series of properties. We typically require that all blocks have the same size k, and that any t points lie in the same number of blocks, usually denoted λ. A design with these parameters is called a $t-(v, k, \lambda)$ design and an abundance of research has been done in constructing designs with different properties (see [1] for a comprehensive summary). In many cases, we require that any *two* points lie in exactly *one* block. A design with these parameters (called a *Steiner system*) is written as a $2-(v, k, 1)$ design. Many examples of such designs occur naturally when one examines the structure of finite fields, finite geometries, and graphs, for instance.

In some situations, it is desirable to partition the set of points into blocks. For instance, if the points represented students in a class, and you wished to partition the students into equally sized groups in order to complete a project. Clearly, this would require that k (the size of the groups) be a factor of v (the total number of students). Now suppose that your students are broken into groups on a particular day, and several days later you wish to break the students into groups again, but subject to the condition that no two students work together a second time. In the language of designs, we wish to find a second collection of $\frac{v}{k}$ blocks that again partition the point set. An instructor might naturally wish to create such partitions over the course of an entire semester.

It is not at all clear how many days an instructor could assign groups before running into problems. Let's look at a small example. Suppose we have six students, labeled 1 through

6, and we wish to break them into three groups, each of size $k = 2$. We refer to the partition used on day i as $\mathcal{P}_i$. Using this notation, one five day schedule could be obtained as follows:

$$\mathcal{P}_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$$
$$\mathcal{P}_2 = \{\{1, 3\}, \{2, 5\}, \{4, 6\}\}$$
$$\mathcal{P}_3 = \{\{1, 4\}, \{2, 6\}, \{3, 5\}\}$$
$$\mathcal{P}_4 = \{\{1, 5\}, \{2, 4\}, \{3, 6\}\}$$
$$\mathcal{P}_5 = \{\{1, 6\}, \{2, 3\}, \{4, 5\}\} .$$

Clearly this schedule cannot be extended since, at this point, student 1 has worked with every other student. Naturally, we say a schedule is *maximal* if it cannot be extended. In general, with groups of size 2, a schedule could never last longer than $v - 1$ days. But is it possible that by arranging groups in some other way, we could find a schedule that lasts fewer than $v - 1$ days, yet still cannot be extended? Consider the following 3-day schedule using the same students:

$$\mathcal{P}_1 = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$$
$$\mathcal{P}_2 = \{\{1, 5\}, \{2, 6\}, \{3, 4\}\}$$
$$\mathcal{P}_3 = \{\{1, 6\}, \{2, 4\}, \{3, 5\}\} .$$

Since students 1, 2, and 3 have each worked with students 4, 5, and 6, any additional partitions would require that students 1, 2, and 3 be paired among themselves. Clearly this is impossible! So, our 3-day schedule above is also maximal. For a group of $n = j \cdot k$ students to be broken into j groups, each of size k, over a period of days, we let $m(j, k)$ denote the minimum length of a schedule that cannot be extended, that is, a minimum length maximal schedule. Our goal in this honors thesis is to find bounds on $m(j, k)$.

One important aspect of this work involves a natural connection to graph theory. A graph is simply a set of points or *vertices*, together with a collection of unordered pairs of vertices, called *edges* (all of the fundamental definitions can be found in [3], for instance). Graphs are often drawn in the plane using dots and lines between pairs of dots to represent the edges. As we wish to examine our problem above in the case when $k = 2$, it is quite natural to model our students with vertices of a graph where edges correspond to the groups of size 2. A partition of the students into j groups of size 2 would correspond to a set of edges $\mathcal{E}$ in the graph such that every vertex is the endpoint of some edge in $\mathcal{E}$ and no two edges of $\mathcal{E}$ share an endpoint. In the language of graph theory, this is called a *perfect matching*, and any collection of edges with no endpoints in common is simply called a *matching*. As we explore bounds on $m(j, 2)$ we will use this language in our graph theoretic models.

2 A solution when $k = 2$

When $k = 2$, we are breaking the students into j distinct groups each of size 2. We wish to determine the value of $m(j, 2)$, for all j. To analyze this problem, we set up a graph theoretic model. For any fixed schedule we define a family of graphs $\{G_i\}$, each of order $2j$, where each vertex represents one of our students and i represents the number of days that have been scheduled successfully. If two vertices represent students that have worked together in a group on any of the i days, we will draw an edge between the corresponding vertices. So, the graphs G_i grow in size as i increases. In fact, the graph G_i contains exactly $i \cdot j$ edges. Each graph G_i is a subgraph of the complete graph $G = K_{2j}$ that models the $2j$ students and all possible pairings. Recall that the complement of the graph G_i, denoted $\overline{G}_i$, is the graph having the same vertex set as G_i, and vertices u and v are adjacent in $\overline{G}_i$ if and only if u and v are not adjacent in G_i. For the following lemmas, we let the family of graphs $\{G_i\}$ correspond to *any* schedule, regardless of its length.

Lemma 2.1. *Let S be a schedule of length i, $1 \leq i \leq 2j - 1$, and let G_i be the corresponding graph that models the ij pairs of students that have worked together during the i days. If the graph $\overline{G}_i$ contains a Hamiltonian cycle, then $m(j, 2) \geq i + 1$.*

Proof. Recall that the edges of the graph G_i represent the pairs of students who have already worked together. Hence, the edges of $\overline{G}_i$ represent the pairs of students who have not yet worked together. As described in the introduction, we will denote each day as a partition $\mathcal{P}_i$ into pairs of vertices of the graph G. Such partitions correspond to perfect matchings in the graph G. Now suppose $\overline{G}_i$ contains a Hamiltonian cycle for some i with $1 \leq i \leq 2j - 1$. We want to find another partition $\mathcal{P}_{i+1}$ in order to extend our schedule by another day. Let $v_1, v_2, v_3, ..., v_{2j}$ represent a Hamiltonian cycle in this graph. Then the partition

$$\{\{v_1, v_2\}, \{v_3, v_4\}, ..., \{v_{n-1}, v_n\}\}$$

could be used for $\mathcal{P}_{i+1}$. Hence, a schedule of length i can always be extended to a schedule of length $i + 1$ provided that the graph $\overline{G}_i$ contains a Hamiltonian cycle. □

We now recall a basic theorem from graph theory concerning Hamiltonian cycles. Dirac's Theorem says that if every vertex in a graph G of order $n \geq 3$ has degree greater than or equal to $\frac{n}{2}$, then G contains a Hamiltonian cycle.

Theorem 2.2. *For all j, $m(j,2) \geq j$.*

Proof. From our lemma we know that if a Hamiltonian cycle exists in $\overline{G}_i$ then there exists another perfect matching in $\overline{G}_i$ which implies that the schedule can be extended another day. For every schedule with corresponding graphs $\{G_i\}$, we show that there exists a Hamiltonian cycle in $\overline{G}_i$ when $i < j$. We know that for every i, $1 \leq i \leq j-1$, every vertex of G_i has degree i. But, since $i \leq j-1$, every vertex of $\overline{G_i}$ has degree at least $(2j-1)-(j-1) = j$. Since the order of $\overline{G}_i$ is $n = 2j$, we see that every vertex of $\overline{G}_i$ has degree at least $\frac{n}{2}$. It follows from Dirac's theorem that $\overline{G}_i$ contains a Hamiltonian cycle and therefore a perfect matching. So every schedule of length i, $1 \leq i \leq j-1$ can be extended another day. Therefore, $m(j,2) \geq j$ for all j. □

The astute reader may observe that if we choose the partition given by the Hamiltonian cycle we will actually be able to add two days to the schedule. We want to keep in mind, however, that we are looking for the minimum number of days in a maximal schedule. By the existence of the Hamiltonian cycle, there exist two additional perfect matchings in the graph. However, there is still a possibility that we can cleverly choose a different partition (not corresponding to the Hamiltonian cycle) so that we can only extend the schedule by one day rather than two. Hence, the existence of a Hamiltonian cycle only *guarantees* us the addition of one more day to our minimum length maximal schedule.

Theorem 2.3. *For j odd, $m(j,2) = j$.*

Proof. From Theorem 2.2 we have that $m(j,2) \geq j$. So to show the equality, we must simply show that $m(j,2) \leq j$. One way to do that would be to show an algorithm that will give us j days but no more. Label the vertices 1 through $2j$ and arrange them in a $2 \times j$ array as follows:

1	2	3	...	j
$j+1$	$j+2$	$j+3$	...	$2j$

Columns in the table represent pairs of people that work together on the initial day. Now, fix the top row and shift the bottom row by one. The following diagrams show the partitions for the first three days.

	1	2	...	j
Day 1	j+1	j+2	...	2j

	1	2	...	j
Day 2	j+2	j+3	...	j+1

	1	2	...	j
Day 3	j+3	j+4	...	j+2

After j days every vertex in the top row will have paired with every vertex in the bottom row. So to find a partition for another day, we would need to pair up all vertices in the top row. Since there is an odd number of vertices in the top row, we see that this is impossible. Thus we have a maximal schedule and we have that $m(j, 2) = j$ for j odd. □

Theorem 2.4. *For j even, $m(j, 2) \leq j + 1$*

Proof. To prove this statement we find an algorithm that will allow us to obtain a maximal schedule after $j + 1$ days. Referring to the $2 \times j$ array as before, let's first fix the first $j - 1$ entries. Take the last entry in the top row and move it to the first position in the bottom row. Shift all entries on the bottom row over by one while moving the last entry in the bottom row up to the last place in the top row. Continuing to shift cyclically the bottom row together with the last entry of the top row gives us a schedule of length $j + 1$ as shown by the following diagrams in the case where $j = 4$:

	1	2	3	4
Day 1	5	6	7	8

	1	2	3	8
Day 2	4	5	6	7

	1	2	3	7
Day 3	8	4	5	6

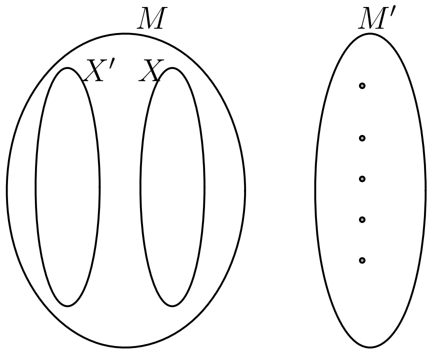
	1	2	3	6
Day 4	7	8	4	5

	1	2	3	5
Day 5	6	7	8	4

.

We now want to see if this schedule is maximal. To create an additional partition, all the vertices in the first $j-1$ positions must pair up. But $j-1$ is odd. Thus no further partitions can be made. □

We now want to argue that $m(j, 2) = j+1$ when j is even. To proceed, let's assume that we have successfully scheduled j days. We can make this assumption since we know that $m(j, 2) \geq j$. Furthermore, suppose we try to arrange a partition to be used on day $j+1$ but are unsuccessful in that we only are able to find a *partial* matching. The following diagram will be useful in the lemmas.



We are showing in this diagram the *partial* matching in $\overline{G}_j$ indicated by M. In other words, all vertices that are paired up in the partial matching are in M. The set M' is the set of vertices that are not in the matching. The set $X \subseteq M$ is the set of all vertices in the graph $\overline{G}_j$ that are neighbors to a vertex in M'. The set X' is all the vertices in M that are not neighbors to any vertex in M'. In addition to these definitions we will also define $\mathcal{E}_m$ to be the set of all edges in the partial matching. Although $\mathcal{E}_m$ is a set of edges, we will refer to vertices as being in the partial matching if and only if they are endpoints to an edge of $\mathcal{E}_m$.

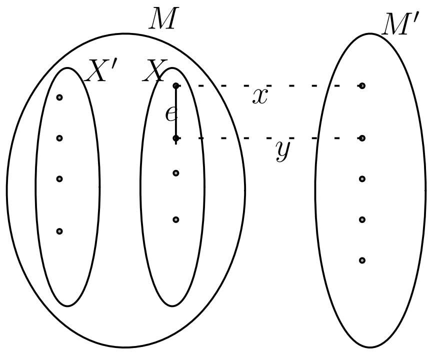
We will first look at the size of the partial matching and then prove that, for j even, we can actually extend the schedule by one more day. We do this by first looking at the case when at least four vertices are in M' (i.e., not part of the partial matching), and then further break into cases based on the size of X. In each case, we show that any partial matching is not maximum, ultimately showing that any partial matching can be used to create a perfect matching and therefore extend the schedule by one more day. The first two cases are slightly easier and follow from the following two lemmas. Note that if any vertex of M' has a neighbor in M' we can extend $\mathcal{E}_m$. So, for the following, we assume no vertex in M' has a neighbor in M'.

Lemma 2.5. *If $|M'| \geq 4$ and $|X| < j-1$, then the matching $\mathcal{E}_m$ is not maximum.*

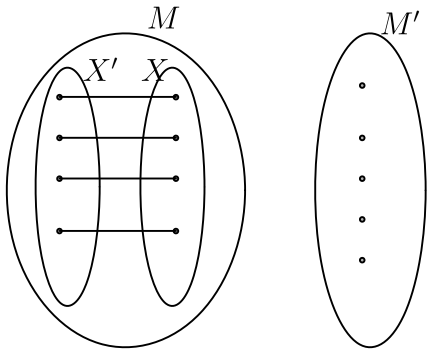
Proof. For this proof we will assume that the matching $\mathcal{E}_m$ is maximum and prove that $|X| > j-1$. We know that every vertex in M' has all of its neighbors in X. We know that every vertex in M' has $j-1$ neighbors. This implies that $|X| \geq j-1$. So by contrapositive $\mathcal{E}_m$ is not maximum. □

Lemma 2.6. *If $|M'| \geq 4$ and $|X| = j-1$, then the matching $\mathcal{E}_m$ is not maximum.*

Proof. If $|X| = j-1$, then every vertex in M' has every vertex of X as a neighbor. We know that $|M| \leq 2j-4$, so we can calculate that $|X'| \leq j-3$. Now we want to find an edge e that lies entirely in X and is an element of $\mathcal{E}_m$. This will allow us to do an edge replacement that will incorporate two new vertices into the matching as shown in the following diagram.



We see in the diagram that if e exists then we can remove it and add the edges x and y. Let us assume that e does not exist. Then all edges in $\mathcal{E}_m$ have one endpoint in X and the other endpoint in X' as pictured below.

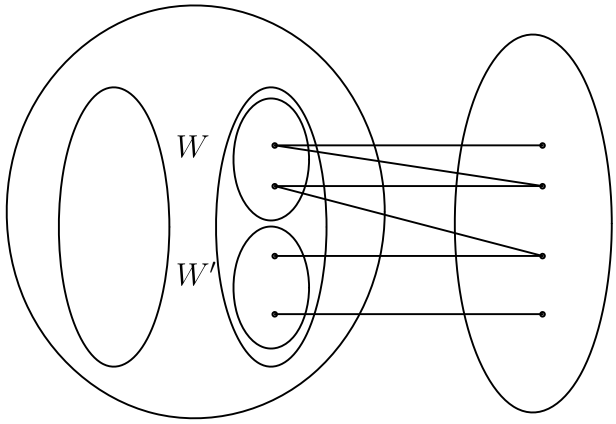


We know that $|X'| \leq j - 3$ and $|X| = j - 1$ so there exist two vertices in X that cannot be endpoints for an edge in the matching. So e must exist. Thus we can do the replacement, and so our partial matching $\mathcal{E}_m$ is not maximum. □

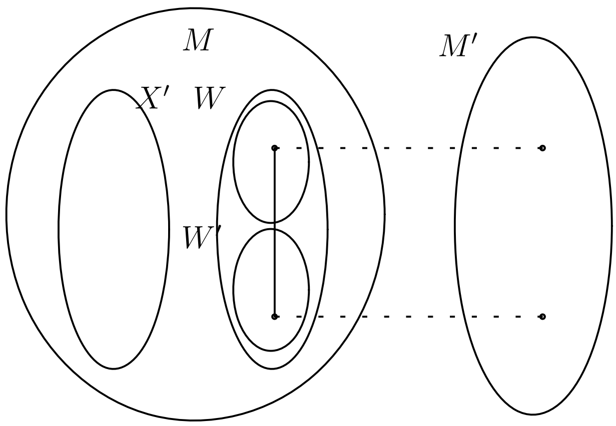
The last possibility is that $|X| > j - 1$ and is slightly more complex. We proceed with the following lemma.

Lemma 2.7. *If* $|M'| \geq 4$ *and* $|X| > j - 1$*, then matching* $\mathcal{E}_m$ *is not maximum.*

Proof. We will need to introduce two new sets of vertices to prove this lemma. We will let the set W be the subset of X containing all vertices that have at least two neighbors in M', and let W' be the set of all vertices in X that have exactly one neighbor in M'. We will use the following diagram to illustrate these sets.



Note here that if there exists an edge of $\mathcal{E}_m$ with both endpoints in X and at least one of the endpoints in W then we are done. In this case we can simply replace black edges with dashed edges and obtain a larger matching as illustrated in the following diagram.



We can do this because if an endpoint v of e is in W, then v has two neighbors in M'. Even if the other endpoint u of e is in W', you can simply allow u to pair with its neighbor in M' and pair v with its different neighbor in M'. So if there exists an edge of $\mathcal{E}_m$ that lies entirely in X and has at least one endpoint in W then we can clearly make a larger partial matching.

We use several inequalities to show that there exists an edge of $\mathcal{E}_m$ with at least one endpoint in W. The idea is to show that the number of edges of $\mathcal{E}_m$ that lie entirely in X must be greater than the number of edges that can exist in W', namely $\frac{|W'|}{2}$. This forces the existence of the edge we desire.

First consider the case that $|W| \geq j-1$. Then, the number of vertices in M that are outside of W is at most $j-3$. Therefore, not every vertex in W can have its neighbor from the matching outside of W. So, some element of $\mathcal{E}_m$ has both of its endpoints in W. Therefore, we can do the replacement as described earlier.

Now assume that $|W| < j-1$. In this case, we let a be the positive integer satisfying $|W| + a = j-1$. We will first note three equations that will help us throughout the proof. From the definitions of W, W', X, and X', it follows that

$$|W| = j - 1 - a, a \in \mathbb{Z}^+, \tag{1}$$

$$|X| = |W| + |W'|, \text{ and} \tag{2}$$

$$|M| - |X| = |X'|. \tag{3}$$

Now we will use these equations to determine an upper bound on the size of X'. Since every vertex in M' has degree $j-1$, we know that if $|W|$ is less than $j-1$, then each vertex in M' must have neighbors in W'. More importantly we can create an inequality using this fact. Since each vertex of M' has at most $j-1-a$ neighbors in W, each vertex of M' has *at least* a neighbors in W'. But no two vertices of W' have a common neighbor in M'. This gives us the inequality $|W'| \geq a|M'|$. Now using this result we find that $a \leq \left\lfloor \frac{|W'|}{|M'|} \right\rfloor$. Now we are able to find an inequality for the size of W, namely

$$|W| \geq j - 1 - \left\lfloor \frac{|W'|}{|M'|} \right\rfloor. \tag{4}$$

It then follows from Equation (2) that

$$|X| \geq j - 1 - \left\lfloor \frac{|W'|}{|M'|} \right\rfloor + |W'|,$$

and therefore, by Equation (3),

$$\begin{aligned} |X'| &\leq |M| - \left(j - 1 - \left\lfloor \frac{|W'|}{|M'|} \right\rfloor + |W'|\right) \\ &\leq (2j-4) - \left(j - 1 - \left\lfloor \frac{|W'|}{|M'|} \right\rfloor + |W'|\right) \\ &\leq j - 3 + \left\lfloor \frac{|W'|}{|M'|} \right\rfloor - |W'|. \end{aligned} \tag{5}$$

Now we note that the number of edges that lie entirely in X must be at least $\frac{|X|-|X'|}{2}$. So if we show that this quantity must be greater than $\frac{|W'|}{2}$ we have proven the statement. To this end, note that

$$\begin{aligned}
|X| - |X'| &\geq \left(j - 1 - \left\lfloor \frac{|W'|}{|M'|} \right\rfloor + |W'|\right) - \left(j - 3 + \left\lfloor \frac{|W'|}{|M'|} \right\rfloor - |W'|\right) \\
&\geq 2\left(1 - \frac{|W'|}{|M'|} + |W'|\right) \\
&\geq 2\left(1 - \frac{|W'|}{4} + |W'|\right). \qquad (6)
\end{aligned}$$

We see that this is clearly greater than $|W'|$ and thus we have shown that some edge in $\mathcal{E}_m$ must have a vertex in $|W|$. So we can make the replacement outlined earlier. □

These lemmas can now be used to prove the main result.

Theorem 2.8. *If a schedule has gone through j days then there exists a partial matching that contains all but 2 vertices for all values of j.*

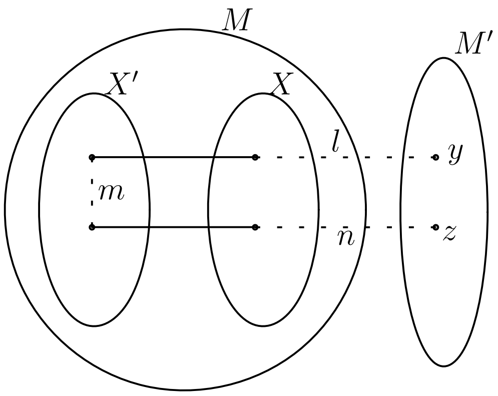
Proof. By Lemma 2.5, 2.6, and 2.7 after j days the corresponding graph $\overline{G}_j$ must have a partial matching that contains all but 2 vertices. □

So to this point we have shown that if a schedule extends over j days, then we can still pair up all but two of the students to create a partition for another day. Note also that our results are true for all values of j. Furthermore, we have already shown that for j odd, $m(j, 2) = j$ and $j \leq m(j, 2) \leq j + 1$ for j even. We will now use Theorem 2.8 to prove that $m(j, 2) > j$ for j even.

To proceed, we suppose that j days have been scheduled and that j is even. Moreover, a partial matching in the graph $\overline{G}_j$ has been found. By the previous lemmas, we can assume that only two vertices of $\overline{G}_j$ are not part of the matching, call them y and z. These two vertices have two sets of neighbors, say Y and Z respectively. Our set W, as defined earlier, is simply $Y \cap Z$, and $W' = (Y \setminus Z) \cup (Z \setminus Y)$.

Lemma 2.9. *In $\overline{G}_j$, j even, if $Y = Z$ then the partial matching can be used to create a perfect matching.*

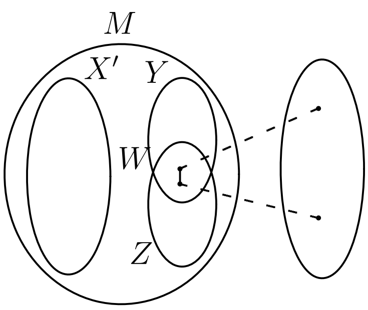
Proof. First note that in this case $Y = Z = X = W$. In other words, the vertices y and z (that are not part of the matching) have the same neighbors. If there exists an edge of the matching with both endpoints in W then, by the conditions we have set already, we would be able to extend the matching by a simple replacement discussed in the previous theorem. So we will assume that all edges in the matching have one endpoint in X and one endpoint in X'. This means that $|X| = |X'| = j - 1$. Now we know that every vertex in the graph has degree $j-1$. Since the vertices of X each have y and z as neighbors, they can have at most $j-3$ neighbors in X'. So, vertices in X' cannot have all of their neighbors in X. This means that there exists an edge, both of whose endpoints are in X'. So we can now make a new replacement by removing the black edges and adding the edges l, m and n to the matching, as pictured below.

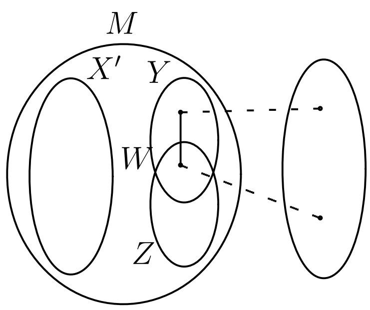


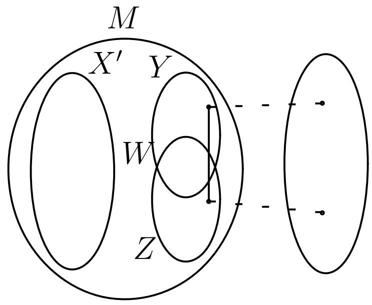
So the schedule can be extended another day. □

Lemma 2.10. *In* $\overline{G}_j$*, j even, if* $0 < |Y \cap Z| < j-1$ *then the partial matching can be used to create a perfect matching.*

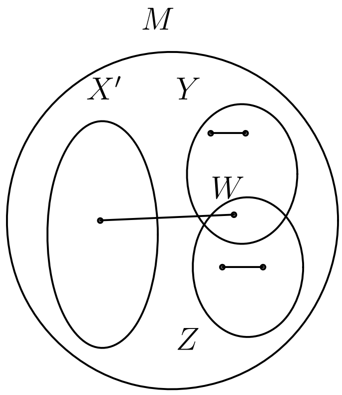
Proof. Since $|X| = |Y \cup Z| > j-1$ in this case, it follows that $|X'| < j-1$. This means that there must be an edge of $\mathcal{E}_m$ both of whose endpoints are in X. We know that if an edge of $\mathcal{E}_m$ has both its endpoints, u and v, in W, then we can do a replacement by adding an edge from u to y and an edge from v to z. If there exists an edge of $\mathcal{E}_m$ with an endpoint u in Y and an endpoint v in W, then we can make a replacement by adding an edge from u to y and an edge from v to z. This also holds true if we have a vertex v in Z and a vertex u in W. Finally, we can also make a similar replacement if there is an edge of $\mathcal{E}_m$ that has an endpoint u in Y and an endpoint v in Z. These replacements are illustrated in the following diagrams.



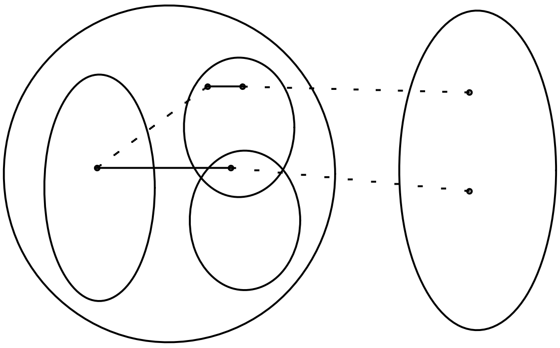




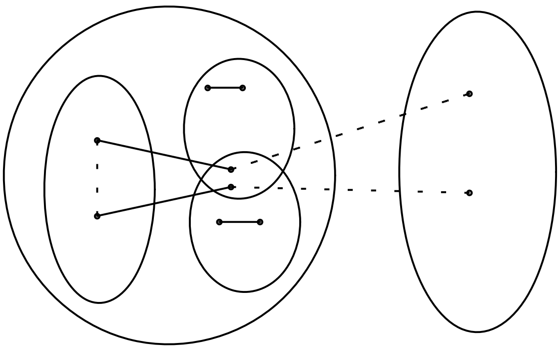
If any of these black edges exist then we simply remove the black edges, add the dashed edges to the matching, and extend the schedule to one more day. So let's assume that no such edge exists in $\mathcal{E}_m$. So if an edge of $\mathcal{E}_m$ is entirely in X, then it lies entirely in $Y \setminus W$ or entirely in $Z \setminus W$. By our conditions and the fact that every vertex has degree $j-1$, it follows that every vertex in W has its neighbor from the partial matching in X'. This gives us the following graph.



Since every vertex has degree $j-1$, we can compute the number of vertices in W. We simply count the number of vertices that lie in Y and not Z. Call this number i. We know that $|Y| = |Z| = j-1$ and $|Y \setminus Z| = i$, it follows by the inclusion/exclusion principle that $|X| = |Y \cup Z| = j-1+i$. Therefore, $|X'| = j-1-i$. Note here that the size of X' is the same as the size of W, and since every vertex in W has a neighbor in X', every vertex in X' has a neighbor in W. So every other edge of $\mathcal{E}_m$ must have both endpoints in Y or both endpoints in Z. Now if there exists an edge between some vertex in Y or Z and some vertex in X' that is not in the matching, then we can extend it to one more day using a simple replacement. We can simply add an edge between the vertex in X' and the vertex in Y, an edge between the vertex in W and z and an edge between the other vertex in Y and y. Similarly we can do the same kind of replacement if there exists an edge between a vertex in Z and a vertex in X'. This is illustrated by the following diagram.



So we will assume that no vertex in X' has a neighbor in Y or Z. So every vertex of X' has all its neighbors in X' or in W. Since the degree of every vertex in X' is $j-1$ and $W \leq j-2$, there must exist some edge not in $\mathcal{E}_m$ that lies entirely in X'. This gives us the following diagram and allows us to make the substitution indicated by the dashed lines.



In every case, we have found another perfect matching and we can extend the schedule by another day. □

Lemma 2.11. *In $\overline{G}_j$, j even, if $|Y \cap Z| = 0$ then the partial matching can be used to create a perfect matching.*

Proof. If $|Y \cap Z| = 0$ then, since $|Y| = |Z| = j-1$, these two sets contain all the vertices in M. We know that if an edge in $\mathcal{E}_m$ has one endpoint in Z and one endpoint in Y then we can extend the matching by a simple substitution. So assume that every edge of the matching is either entirely in Y or entirely in Z. Since $j-1$ is an odd number this is impossible. Thus we must have some edge that has an endpoint in Z and an endpoint in Y, and we can create another perfect matching. □

Combining all of these results we have shown that for j even a schedule of length j can always be extended by one more day. Theorem 2.4 says that $m(j, 2) \leq j+1$ for j even. We utilize this fact and the preceding lemmas to state our main result.

Theorem 2.12. *If j is even, then $m(j, 2) = j + 1$.*

3 Conclusion

The problem discussed here arises very naturally as it is typical to form schedules using these constraints. We found that for the special case when $k = 2$ that $m(j, 2) = j$ for j odd, and $m(j, 2) = j + 1$ for j even. This leads to the obvious question of what happens when we change the value of k. The problem becomes much more complex but potentially gives rise to some very interesting mathematics. For example, when we look at the case $m(j, 3)$, we can form upper bounds when j is prime by using distinct primitive roots to cyclically shift each row as described earlier. If, however, j is not prime, then we may run into problems finding primitive roots and thus have difficulty finding an upper bound on $m(j, 3)$. In any case, there is potential for much more research to be done in determining both upper and lower bounds on $m(j, k)$, $k > 2$.

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